Dynamic phase transition in a kinetically constrained model for traffic

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We address the emergence of traffic jams using dynamic correlation functions in analogy to kinetically constrained models for glasses. In kinetically constrained models, the formation of glass becomes a true (singular) phase transition in the limit $T \to 0$. Similarly, using the Nagel-Schreckenberg model to simulate traffic flow, we find that the emergence of jammed traffic acquires the signature of a sharp transition in the deterministic limit $p \to 1$, corresponding to overcautious driving. We identify for the first time a true dynamical critical point marking the onset of coexistence between free flowing and jammed traffic. We find diverging correlations analogous to those at a critical point of thermodynamic phase transitions.

PACS numbers: 64.70.kj, 64.70, 64.70.qj, 89.40.Bb

Dynamic arrest, the sudden slow-down of dynamical systems with increasing density or interaction potential, is a central phenomenon in complex systems across biology, geology, material science, transport and traffic. While the dynamic arrest is important for material stability and memory, it is rather detrimental in traffic or transport, where congestion freezes any motion. A well-known example of dynamical arrest is the sharp increase of the viscosity of glass forming liquids at the glass transition. Many more systems exhibit dynamic arrest phenomena that are seemingly similar. Examples include dynamics in the crowded environment of cells or human dynamics in escape situations or traffic flows.

While the nature of this transition is not clear, a possible idea is that of a dynamic phase transition, in which the dynamic slow down is considered to be analogous to equilibrium phase transitions with their singularities in thermodynamic quantities. An important question then concerns the universality of the dynamic arrest. While dynamic arrest phenomena across different systems seem to be related, it is not clear to which extent they really are.

Important insight into the dynamic arrest of glasses comes from kinetically constrained models (KCMs), a class of discrete models with stochastic dynamics that are used to describe the glassy behavior and increasing relaxation time scales in supercooled liquids [1]. As the defining ingredient, KCMs have a kinetic constraint that allows local activity only if a local condition is met. These models provide some evidence that indeed the dynamic slow-down is the manifestation of a dynamic critical point [2, 3] in the limit $T \to 0$.

A constraint analogous to those in KCMs exists in traffic flow: cars can accelerate only if the distance to the car in front is sufficiently large. A well-studied model that incorporates a number of basic dynamical properties of real traffic is the Nagel-Schreckenberg (NS) model [4], a lattice-gas-like model with discrete position, time, and velocity. It allows for stochastic fluctuations in the velocities of the individual cars, controlled by the probability p that reflects the drivers' individual freedom to adjust their speed. The NS model describes the formation of traffic jams, and there has been much discussion regarding the (non)existence of a sharp phase transition between free-flowing traffic and traffic with jams [5].

We reserve the word dynamical phase transitions here to denote qualitative changes of behavior, accompanied with singularities in various expectation values in the stationary state. This is analogous to the singularities of observables at thermodynamic phase transitions, with the equilibrium state replaced by the stationary state. While different types of traffic flow are often referred to as "phases" Ref. [6, 7], the traffic jams have a typical size, and there is no divergence of correlation lengths [8], as one expects to be at a phase transition. Consequently, the evidence of dynamical criticality and the connection to other dynamically arrested systems remains elusive. However, a sharp transition without diverging correlations was found in the limit $p \to 0$ [9, 10].

In this Letter, we show that in the deterministic limit $p \to 1$, the simple one-dimensional NS model [4] of traffic flow does exhibit a true dynamical phase transition. We apply concepts normally used for glasses, such as dynamic correlation functions and susceptibilities [2] to show that in the deterministic limit, the maximum correlation length and time diverge, analogous to the dynamic transition in KCMs at $T \to 0$. These relatively simple stochastic dynamical systems allow for a direct comparison of dynamic arrest in different dimensions. We identify observables in traffic which behave similarly and which relate to each other in the same way as in glasses. This signals some degree of universality in dynamic arrest.

The NS model simulates traffic flow on discrete space and time. A fixed number of cars with average density ρ per lattice site are positioned on a one-dimensional lattice with periodic boundary conditions. The cars have integer velocities v_i between 0 and some maximum velocity $v_{\max}$. The dynamics is given by the following update rules applied in parallel to all cars: All cars i with velocity $v_i < v_{\max}$ accelerate by 1. Next, the kinetic constraint is applied, so that, if the distance of car i to the next car

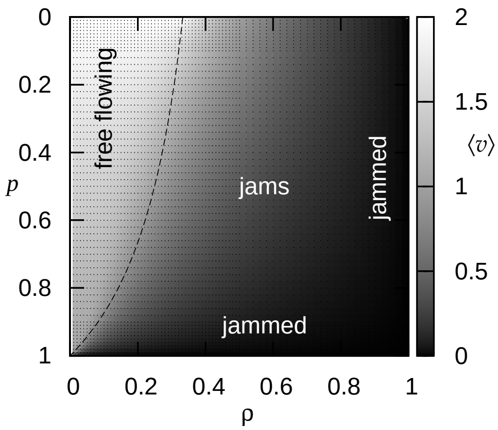

FIG. 1: Phase diagram of the Nagel-Schreckenberg model showing $\langle v \rangle$ for the case of $v_{\max} = 2$ (see gray scale on the right). The dashed line marks the transition between freely flowing traffic and traffic with jams found in Eq. (1).

$d_i < v_i$, then car i decelerates to d_i. With probability p, a car reduces its velocity by 1. This parameter is the only source of stochasticity in the system and controls the fluctuations of the velocities of cars. Finally, all cars move along the road by v_i lattice sites.

We give an overview of the average velocity $\langle v \rangle$ as a function of density ρ and probability p in Fig. 1. For simplicity, we have taken $v_{\max} = 2$. At low density, traffic can flow freely, and $\langle v \rangle = v_{\rm f} \equiv v_{\max} - p$. With increasing density, cars interact and decelerate according to the kinetic constraint, leading to the formation of jams and a decrease in $\langle v \rangle$. We define the transition density $\rho_{\rm tra}$ from the balance of the outflow and inflow rates of a jam, as required for its stability. The outflow rate of a continuous sequence of jammed cars is $1-p$, corresponding to the probability of acceleration of the car at the head of the jam. Because cars approach the rear of a jam with average velocity $v_{\rm f} = v_{\max} - p$, and the rear of the jam itself travels backwards at a speed equal to the inflow/outflow rate, this yields the transition density

$$\rho_{\rm tra} = \frac{1-p}{v_{\max} + 1 - 2p} \ . \tag{1}$$

Consequently, $\rho_{\rm tra} \propto 1-p$ for small $1-p$.

For glasses and KCMs [3, 11, 12], it has been shown that the transition becomes singular at $T = 0$, where the dynamics become deterministic. The question is then whether a similar singular transition exists in a simple one-dimensional model of traffic flow. To explore this analogy, we relate the stochasticity parameter p in the NS model to the temperature T of spin glasses. The case $T = 0$, where the dynamics of glasses freezes entirely, corresponds to the case $p = 1$, where cars always decelerate, and traffic flow arrests. In the limit of $T \to 0$ and $p \to 1$, the systems become deterministic. For real traffic flow, the case $p \to 1$ is rather exceptional, but can be approached in the case of bad weather conditions, where drivers act carefully, and are likely to overreact with braking.

We investigate the onset of jamming at $\rho \sim \rho_{\rm tra}$ by applying dynamic correlation functions that have been much used to investigate the dynamic arrest of glasses. A characteristic property of glasses is their dynamic heterogeneity. Dynamically active regions separate from dynamically less active regions in space and time, leading to increasing dynamic heterogeneity of the system. This dynamic heterogeneity can be quantified by the dynamic susceptibility [2, 13, 14]. We define the dynamic correlation function of traffic flow using

$$G_4(i,t) = \langle c(i;t)c(0;t) \rangle - \langle c(0;t) \rangle^2 \ , \tag{2}$$

where we take the mobility $c(i;t)$ of car i as $c(i;t) = (1/(t+1)) \sum_{t'=0}^{t} v_i(t')$, its average velocity during the time interval $[0,t]$. We find that near $p = 1$, dynamic correlations become indeed increasingly long-ranged when the density approaches $\rho_{\rm tra}$, the onset of the jammed regime. To investigate whether there is a true diverging length scale, we define the dynamic susceptibility

$$\chi_4(t) = \frac{1}{\langle v^2 \rangle - \langle v \rangle^2} \sum_{i=0}^{N-1} G_4(i;t) \tag{3}$$

that measures the number of cars that move cooperatively on the time scale t. The dynamic susceptibility χ_4 indicates the size of regions of correlated mobility, and has been much used to measure dynamic heterogeneity in glasses and granular materials [15–18].

In glasses, maximum cooperative motion arises at intermediate time scales, at which the particles escape their cages. In traffic, cars cannot escape the crowding of their environment independently from each other, and the maximum dynamic susceptibility arises at the shortest time interval, $t = 1$, see inset of Fig. 2a. We use the maximum value of χ_4 to explore dynamic correlations for a range of p-values in Fig. 2(a). Indeed increasing maxima develop at $\rho \sim \rho_{\rm tra}$ as p approaches unity, indicating increasing dynamic correlations. The divergence of the dynamic susceptibility is clearly seen in Fig. 2b, where we plot the maximum value of χ_4 as a function of $1-p$. The figure shows data for various $v_{\max}$; in all cases, the maximum χ_4 grows as a power law $\chi_{4,\max} = (1-p)^{-\nu}$, indicating that the number of cars that move cooperatively diverges. The divergence of the dynamic susceptibility indicates that traffic flow in the NS model becomes truly critical in the limit $p \to 1$. In this model, the exponent of the power law appears to increase weakly with $v_{\max}$ changing from $\nu = 0.53$ to $\nu > 0.70$. We also note that no maximum of χ_4 is observed for $p \to 0$, indicating the absence of dynamic correlations, and the very different nature of this limit. The criticality for $p \to 1$, however, is analogous to the one observed in KCMs at $T \to 0$, and

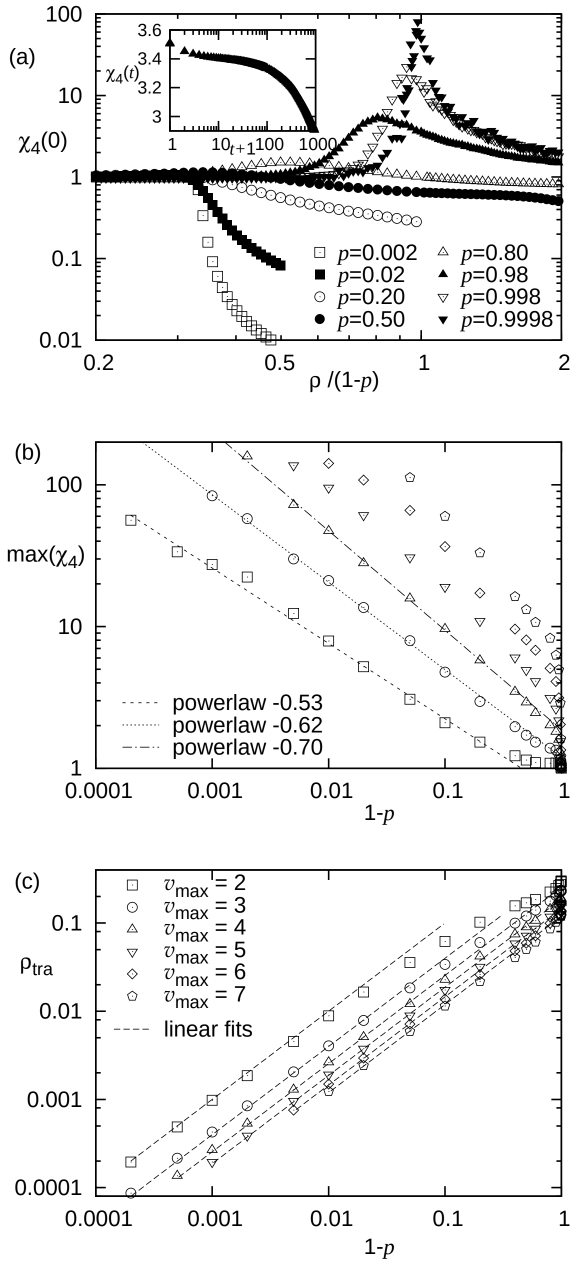

FIG. 2: Dynamic susceptibility of traffic flow in the NS model. (a) The value of $\chi_4(0)$ as a function of rescaled density for a range of probabilities p at $v_{\max} = 2$. The peak sharpens markedly as $p \to 1$. Inset: $\chi_4(t)$ as a function of time shows that the largest value occurs at $t = 0$. (b) Maximum value of $\chi_4(t = 0)$ as a function of the rescaled density, plotted vs. $1 - p$, for a set of $v_{\max}$ (symbols explained in (c)). Power-law behavior (dashed lines) indicates the divergence of the dynamic susceptibility on approach of the critical point $p = 1$. (c) Density of maximum dynamic susceptibility as a function of $1 - p$ for various values of $v_{\max}$. The position of the maximum is well described by the limiting behavior of (1), $\rho_{\mathrm{tra}} = (1 - p)/(v_{\max} - 1)$ indicated by the dashed lines.

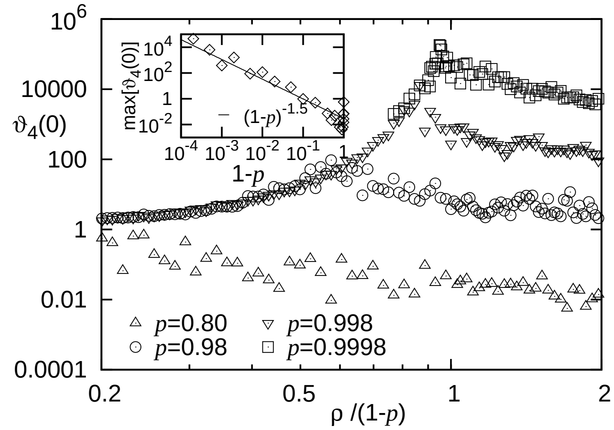

FIG. 3: Residence time of cars in jams as a function of reduced density for various values of p at $v_{\max} = 2$. As $p \to 1$, the correlation time becomes more sharply peaked. Inset: Maximum residence time as a function of $1 - p$ shows power-law divergence of the correlation time for $p \to 1$.

indicates that the deterministic limits $p \to 1$ and $T \to 0$, are dynamical critical points of the systems. The divergence occurs at the onset of the jammed regime; to show this, in Fig. 2(c), we compare the location of the maximum of χ_4 (symbols) with the limiting ($p \to 1$) behavior of ρ_{tra} according to Eq. 1 (dashes lines).

Further evidence of critical behavior comes from measurement of the correlation *time* scale. To estimate the typical time scale, we make use of a quantity similar to χ_4, where we interchange time and car index in the definition of $c(i; t)$ and in the sum appearing in (3), to obtain $\vartheta_4(i)$. The temporal susceptibility ϑ_4 indicates the correlation time scale of the system, and measures the typical residence time of a car in a jam. We plot this correlation time as a function of reduced density in Fig. 3. A strong increase of the maximum of ϑ_4 suggests that in addition to the divergence of the correlation length, there is also a divergence of the correlation *time* scale. This is confirmed by plotting the maximum values of ϑ_4 as a function of $(1 - p)$ in the inset. Similar to the spatial correlations, the correlation time scale diverges as a power law $\vartheta_{4,\max} = (1 - p)^{-\mu}$ as $p \to 1$, confirming that the system behaves critically along the time dimension. We determine the exponent to be $\mu \sim 1.5$. For real traffic, such diverging correlation time can have fatal consequences, as it indicates diverging persistence times of traffic jams. There are historical examples of traffic jams lasting several days (see, for instance, Ref. [19]), under particular weather conditions.

We thus find a dynamical critical point with diverging length and time scales. We note that the condition of densities large enough to cause jams, but not large enough to jam the whole road over its entire length, is analogous to the coexistence of two phases. Indeed, in

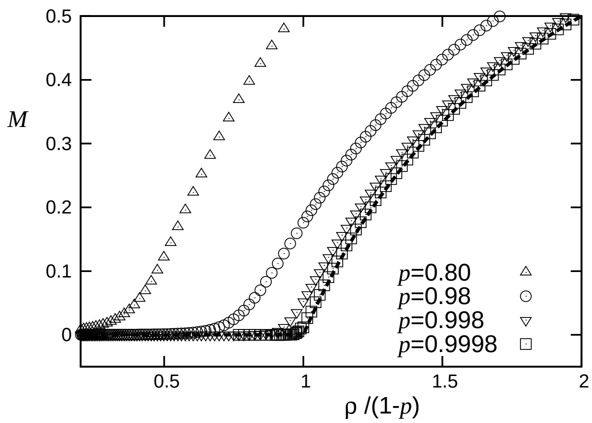

FIG. 4: Order parameter M as a function of reduced density for various values of p at $v_{\max} = 2$. As $p \to 1$, the transition at $\rho = 1 - p$ becomes singular. The dashed line shows the theoretical prediction $M = (\rho - \rho_{\mathrm{tra}})/\rho$.

the limit $p \to 1$, jams coalesce in time to form a single jammed phase coexisting with a single free flowing phase. Taking into account the divergence of the quantites χ_4 and ϑ_4 on entry of the coexistence regime, the situation appears similar to a thermodynamical critical point terminating a two-phase regime.

To explore this analogy further, we define a dynamical order parameter

$$M = \frac{v_{\mathrm{f}} - \langle v \rangle}{v_{\mathrm{f}}} \ , \tag{4}$$

in terms of the average velocity $\langle v \rangle$ and the mean velocity in free flow $v_{\mathrm{f}} = v_{\max} - p$. We show M as a function of the rescaled density in Fig. 4. It is continuous at the transition $\rho = \rho_{\mathrm{tra}}$, but exhibits an increasingly sharp kink as $p \to 1$, indicating a singular point in the limit $p \to 1$. If we assume simple coexistence in the two-phase regime, we can predict the function $M(\rho) = (\rho - \rho_{\mathrm{tra}})/\rho$. Indeed as $p \to 1$, there is strong evidence that the data converge to this simple function, supporting our picture of jam and free flowing traffic as coexisting phases. The functional dependence of this order parameter has an exponent $\beta = 1$, corresponding to a Bose condensate, and to condensates found in typical zero range models [20].

We have shown that a simple 1D model for traffic flow exhibits signature of dynamic arrest analogous to that of kinetically constrained models for glasses. Exploiting this analogy, we have identified and analyzed a non-equilibrium phase transition in the Nagel-Schreckenberg model in the limit $p \to 1$ and $\rho \to 0$, between a free flowing state and coexistence of jammed and free flowing traffic. The hallmark of this transition is the divergence of both correlation length and time scales, giving it a second order character. Defining a proper dynamic order parameter, we have shown that this transition is analogous to a thermodynamic phase transition in which the coexistence regime is entered through the critical point. Finally, the direct analogy to KCMs of glasses points out the universality of dynamic arrest phenomena in systems of different dimensionality.

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